The nuclear shell-model: from single-particle motion to collective effects

1. Nuclear forces and very light nuclei

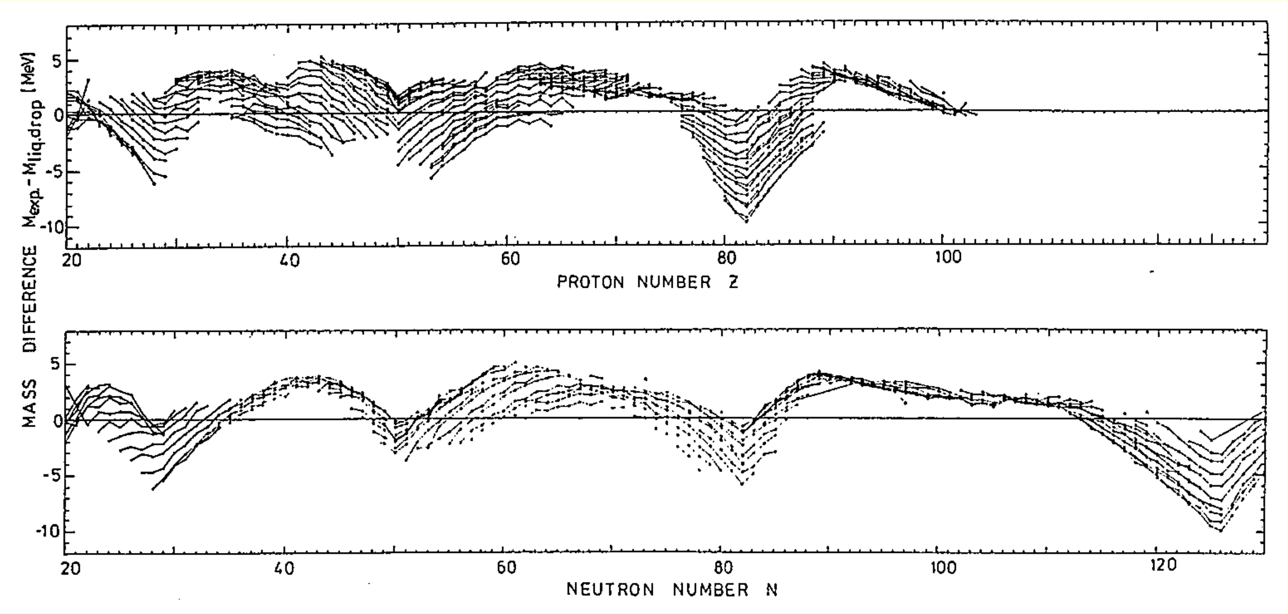
2. Independent-particle shell model and few nucleon correlations

3. Many-nucleon correlations: collective excitations and symmetries

- Magic numbers (stability)
  N,Z = 2, 8, 20, 28, 50, 82, 126

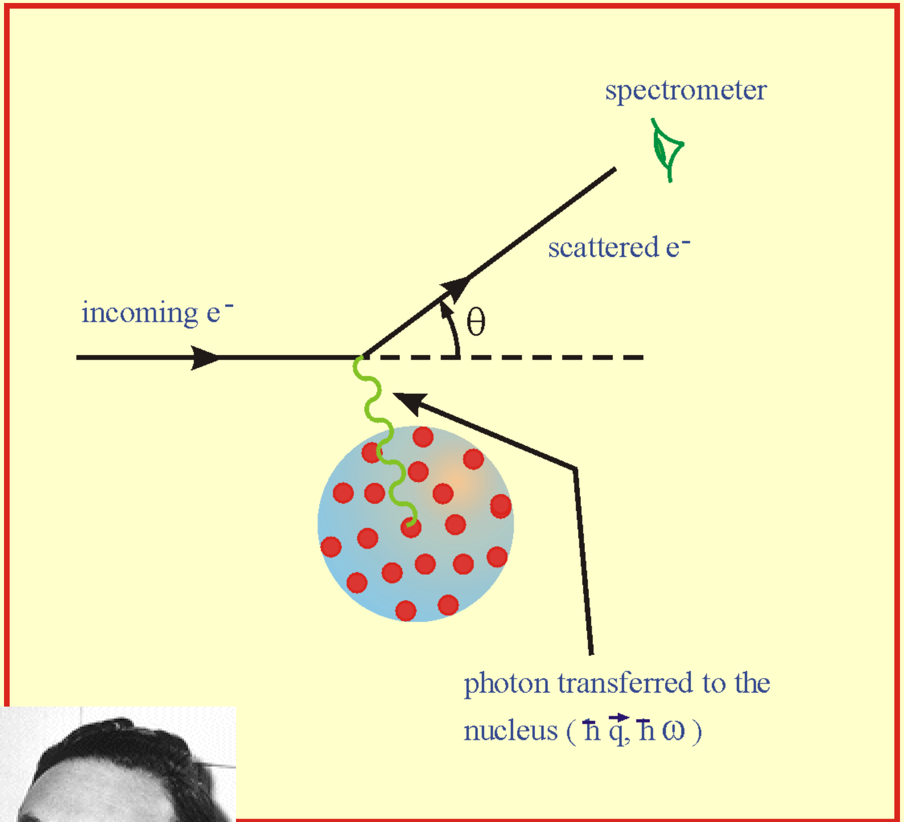
- Spin and parity of ground states of most odd-A nuclei

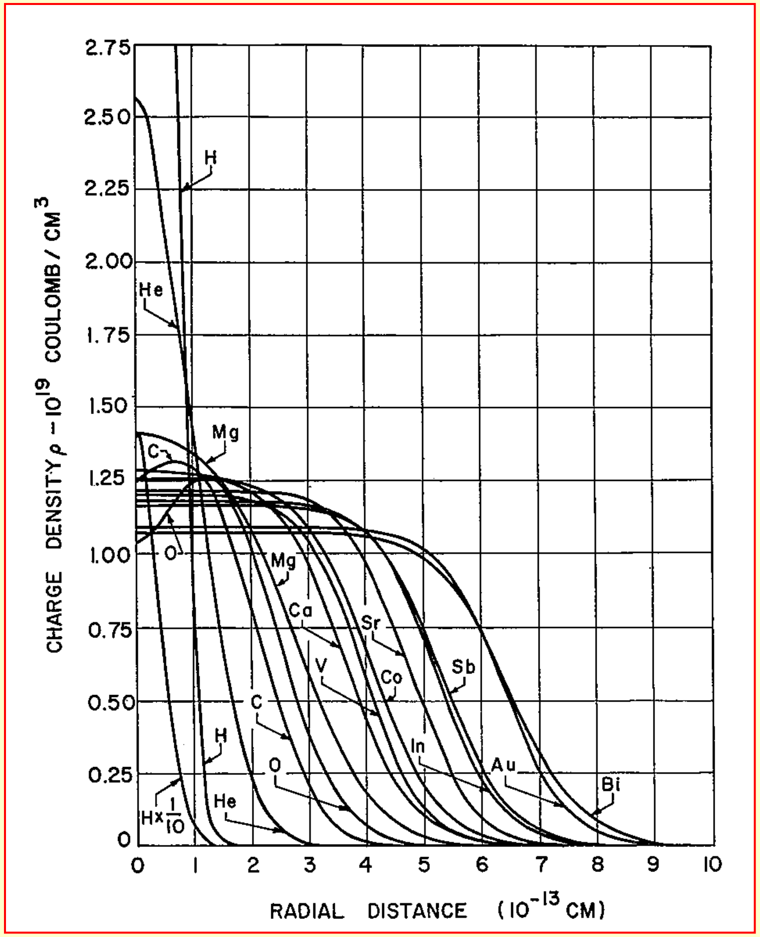
- Magnetic moments of g.s. of most odd-A nuclei



Bethe and Bacher (1936)
von Weisszäcker (1936)

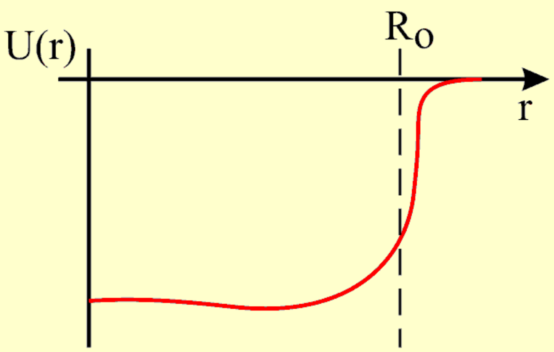
Robert Hofstadter – Nobel prize in physics 1961

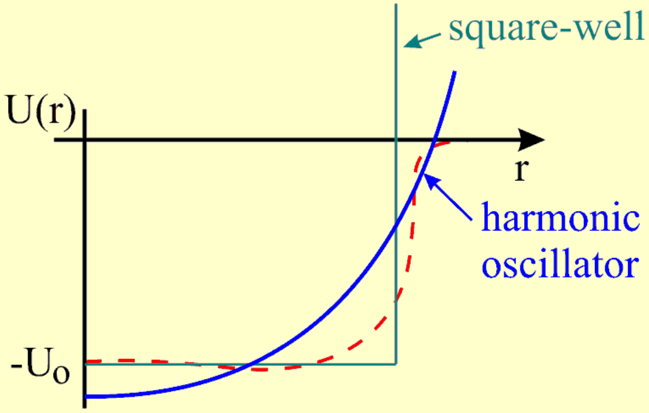




"For his pioneering studies of electron scattering in atomic nuclei and for his thereby achieved discoveries concering the structure of the nucleons" (work in the period 1953-1956 at Stanford-SLAC).

Central problem: choice of U(r)





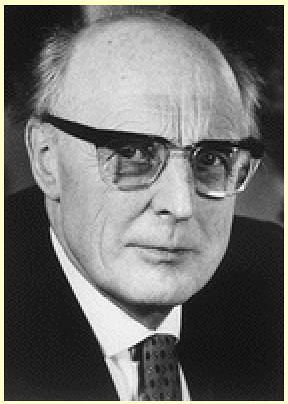
$U(r) = -U_0 \; (r \lessgtr R_0)$
$U(r) = \frac{1}{2} m\omega^2 r^2$ } analytical solutions

Solve central problem Schrödinger eq. $[\frac{\vec{p}^2}{2m} + U(r)]\,\varphi(\vec{r},\vec{s}) = \varepsilon\varphi(\vec{r},\vec{s})$

$\varphi(\vec{r},\vec{s}) = R_{n\ell}(r) Y_\ell^m(\theta,\varphi)\chi_{\frac{1}{2}}^{m_s}(\sigma)$

Harmonic oscillator: $(N+{}^3\!/_2)\,\hbar w$ with $N=2(n-1)+\ell$

n: radial q.n.; ℓ: orbital angular momentum q.n.



Idea of shell-model:

- M. Goeppert-Mayer (1949)
- H. Jensen, O. Haxel and H. E. Suess (1949)

$$U(r)=\frac{1}{2}m\omega^2 r^2+\alpha\,\vec{\ell}.\vec{\ell}+\beta\,\vec{\ell}.\vec{s}$$

On Closed Shells in Nuclei. II

MARIA GOEPPERT MAYER
Argonne National Laboratory and Department of Physics, University of Chicago, Chicago, Illinois
February 4, 1949

THE spins and magnetic moments of the even-odd nuclei have been used by Feenberg[1,2] and Nordheim[3] to determine the angular momentum of the eigenfunction of the odd particle. The tabulations given by them indicate that spin orbit coupling favors the state of higher total angular momentum. If strong spin-orbit coupling, increasing with angular momentum, is assumed, a level assignment different from either Feenberg or Nordheim is obtained. This assignment encounters a very few contradictions with experimental facts and requires no major crossing of the levels from those of a square well potential. The magic numbers 50, 82, and 126 occur at the place of the spin-orbit splitting of levels of high angular momentum.

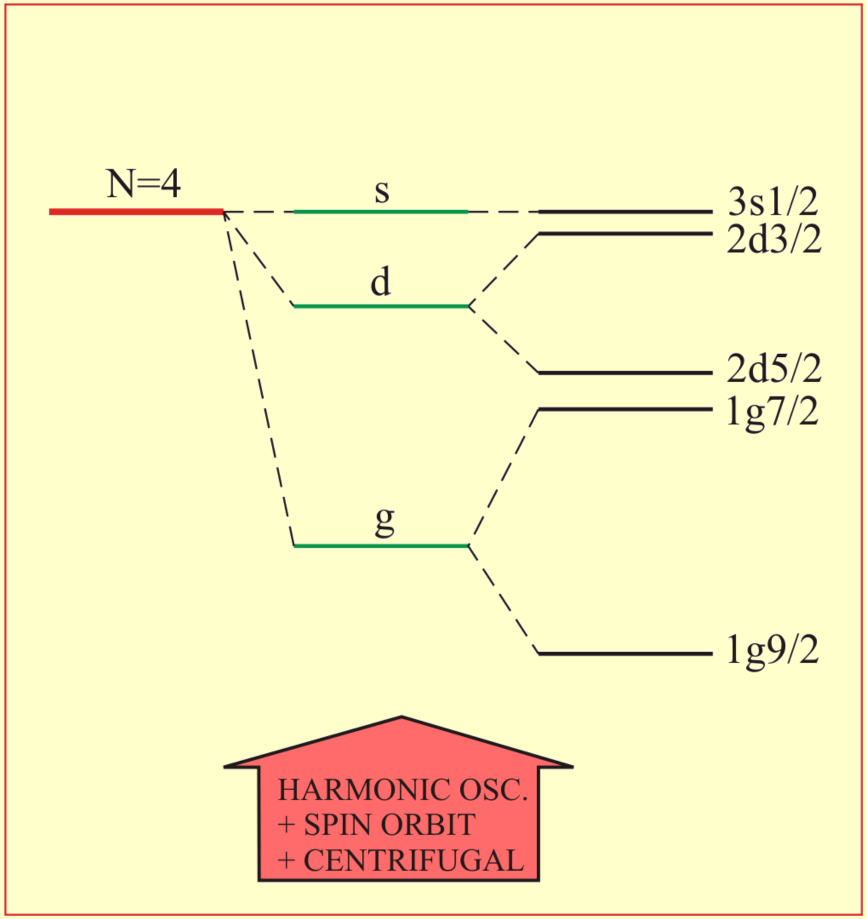
Phys.Rev.75,1969 (1949)

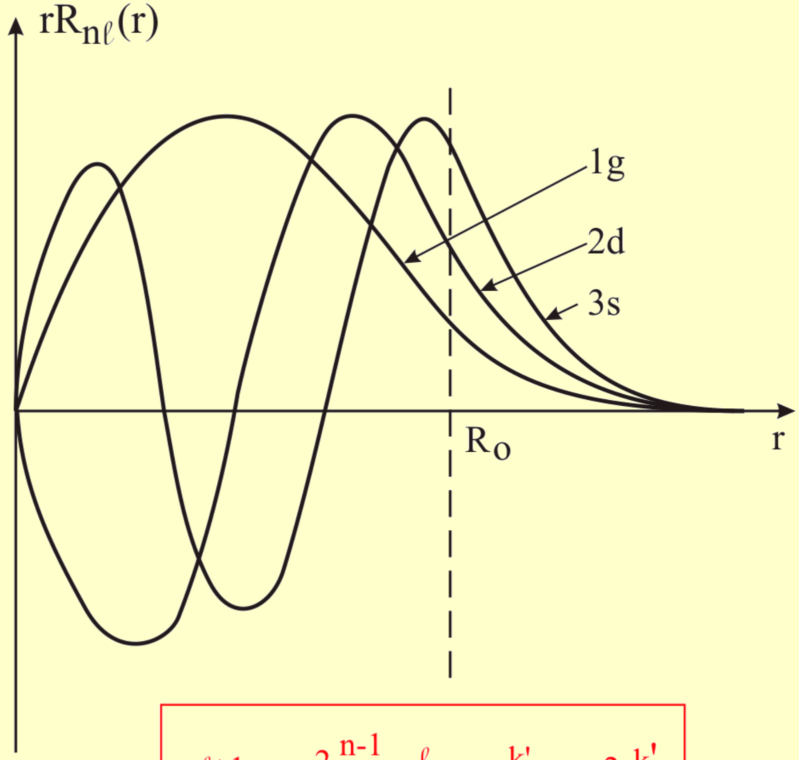
On the "Magic Numbers" in Nuclear Structure

OTTO HAXEL
Max Planck Institut, Göttingen
J. HANS D. JENSEN
Institut f. theor. Physik, Heidelberg
AND
HANS E. SUESS
Inst. f. phys. Chemie, Hamburg
April 18, 1949

A SIMPLE explanation of the "magic numbers" 14, 28, 50, 82, 126 follows at once from the oscillator model of the nucleus,[1] if one assumes that the spin-orbit coupling in the Yukawa field theory of nuclear forces leads to a strong splitting of a term with angular momentum l into two distinct terms $j=l\pm\frac{1}{2}$.

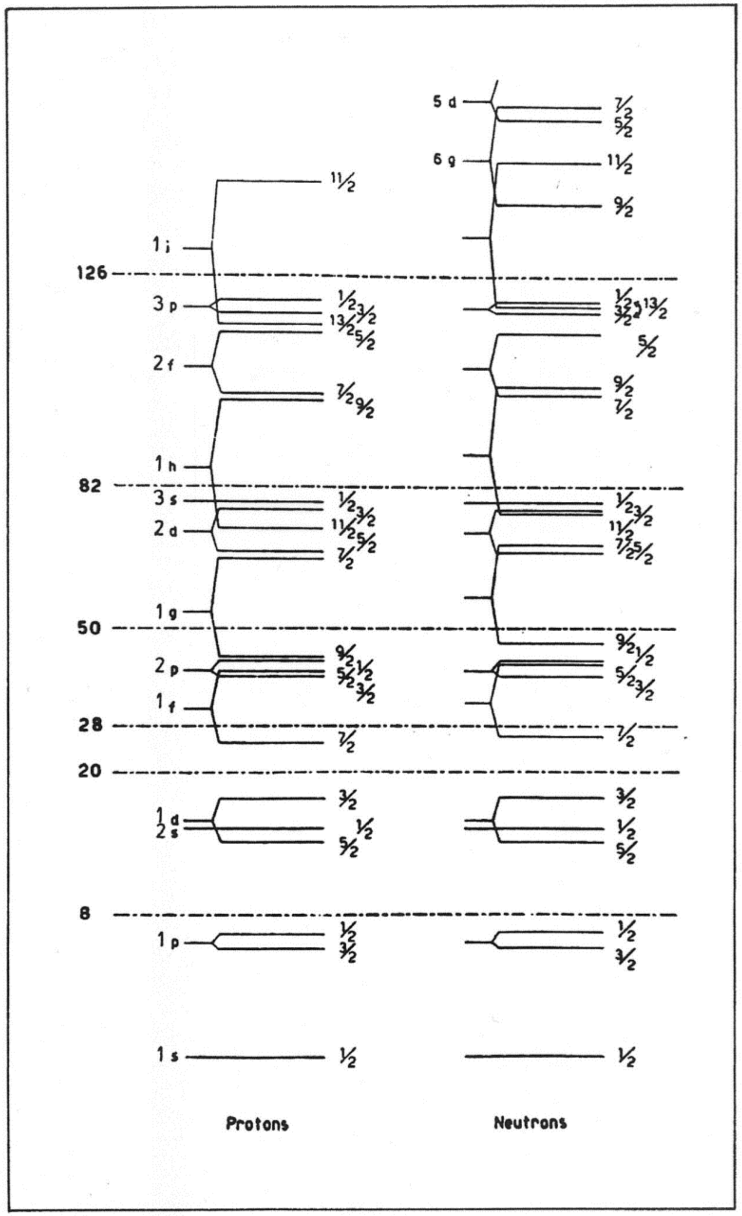
Phys.Rev.75,1766(1949)

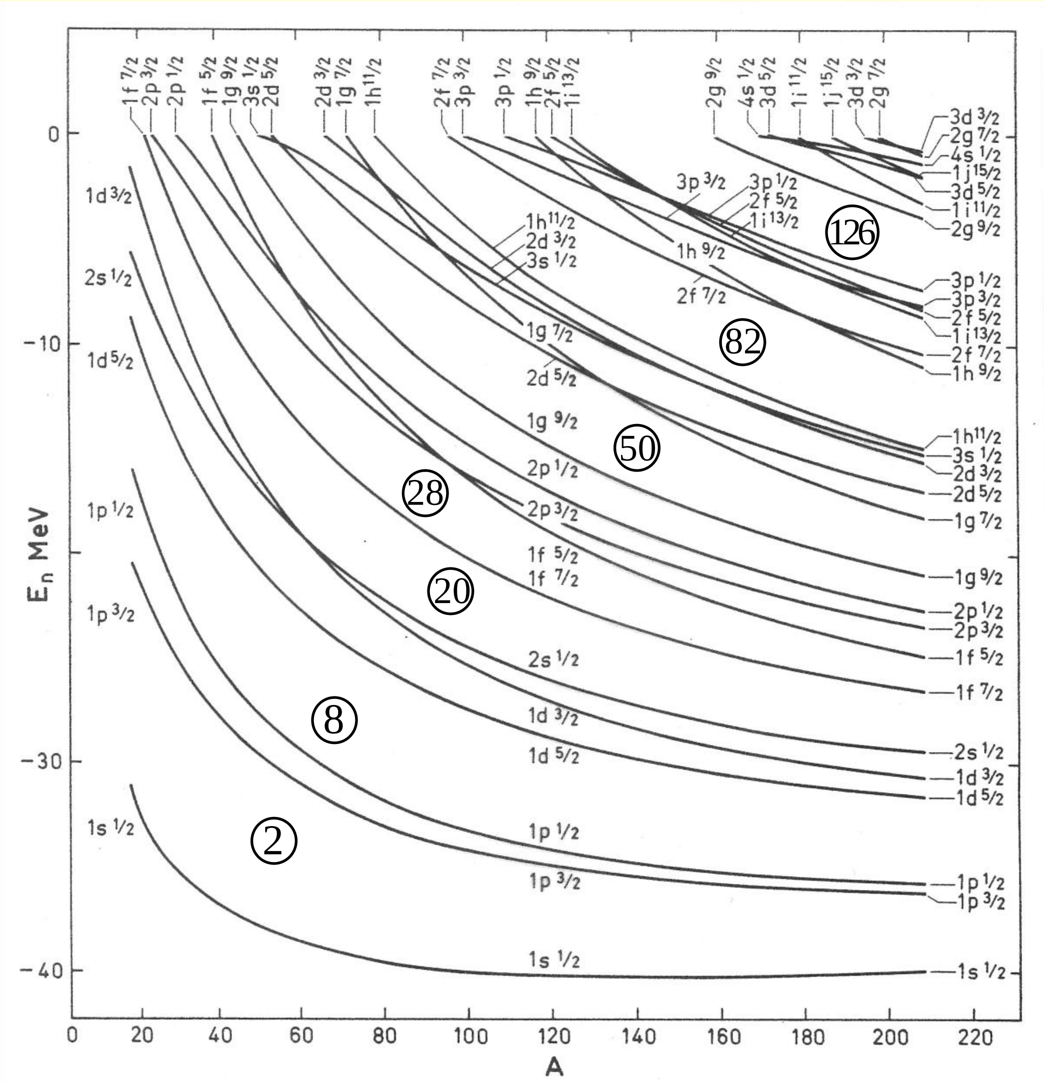




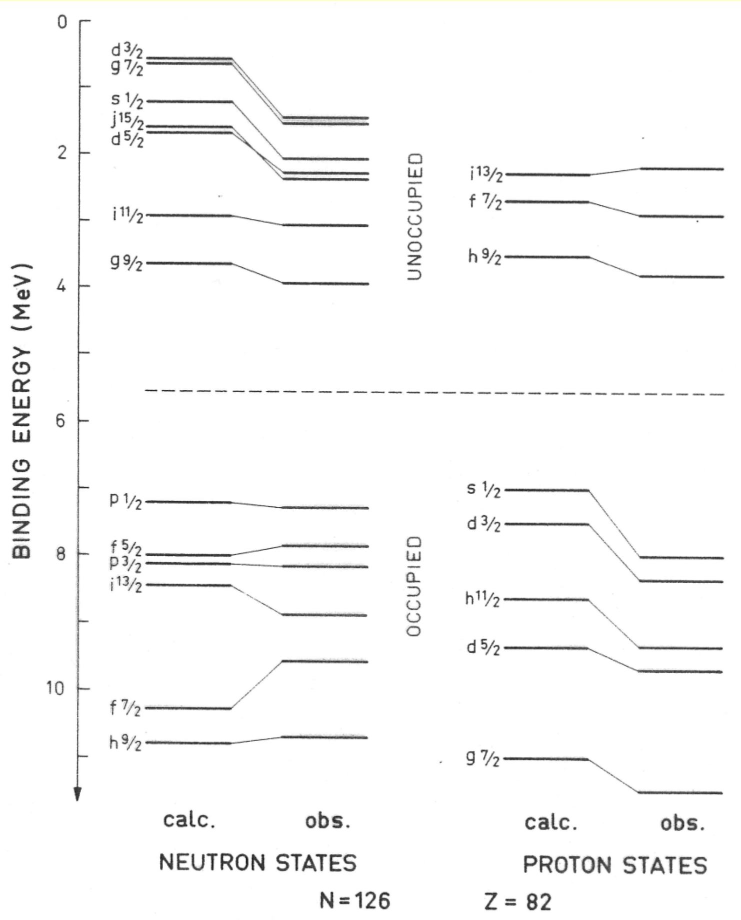
$$r^{\ell+1}.e^{-\nu r^2}\sum_{k'=0}^{n-1} a_{k'}^{\ell}(-1)^{k'}(2\nu r^2)^{k'}$$

7209


5 d
6 g
1 i
126
3 p
2 f
1 h
82
3 s
2 d
1 g
50
2 p
1 f
28
20
1 d
2 s
8
1 p
1 s
Protons
Neutrons



E_n MeV
A
2
8
20
28
50
82
126



BINDING ENERGY (MeV)
UNOCCUPIED
OCCUPIED
calc.
obs.
calc.
obs.
NEUTRON STATES
PROTON STATES
N=126
Z = 82

Construction of basis wave functions

- $Y_l^{m_l}(\theta,\varphi)$ and $\chi^{1/2}_{m_s}$

- $\varphi_{n,l,j,m}(\vec{r},\sigma) = R_{nl}(r)\underbrace{[Y_l \otimes \chi^{1/2}]^j_m}_{\text{ang. momentum coupling}}$

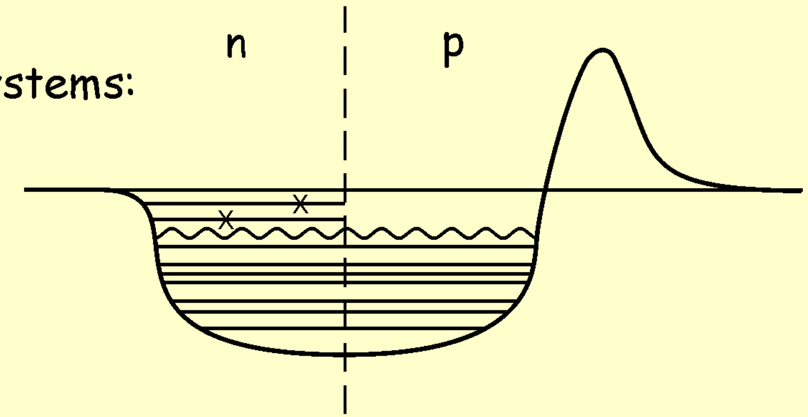
$$: \sum_{m_l,m_s} \langle l\, m_l, 1/2\, m_s | jm \rangle Y_l^{m_l} \chi^{1/2}_{m_s}$$

- Two-particle wave functions $(j_a,j_b)J$

$\psi(j_a j_b, JM) \Rightarrow J$ all values $|j_a - j_b| \leq J \leq j_a + j_b$

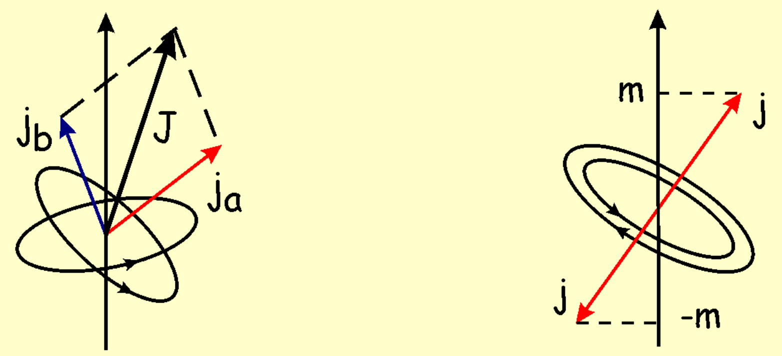
$\psi(j^2, JM) \Rightarrow$ J: 0,2,4,6,.... (even only)

Two nucleon systems:



$$H = h_o(1) + h_o(2) + V(1,2)$$

$$E(j_a j_b, JM) = \varepsilon_{ja} + \varepsilon_{jb} + \langle j_a j_b, JM | V(1,2) | j_a j_b, JM \rangle$$



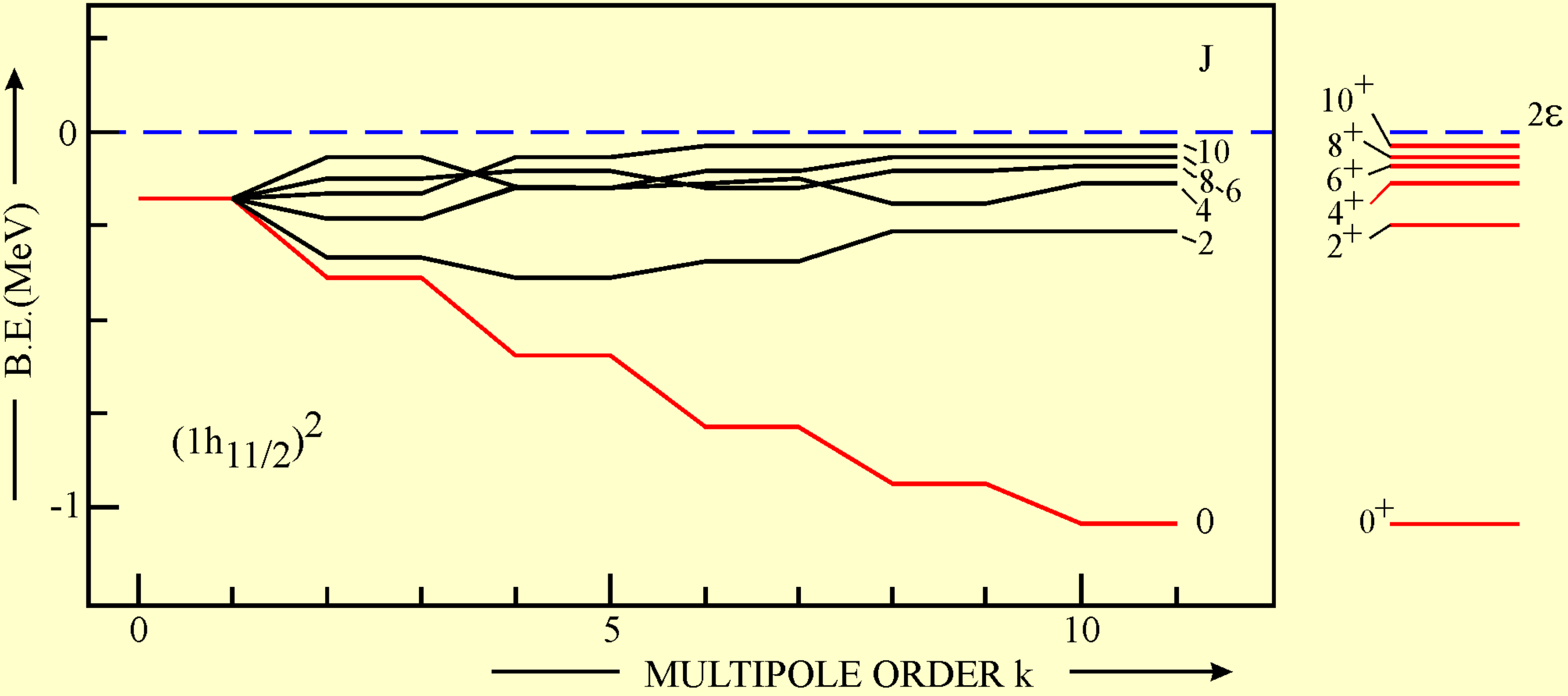
$|j_a - j_b| \leq J \leq j_a + j_b$ $\qquad$ $J = 0, 2, 4, \ldots 2j-1$

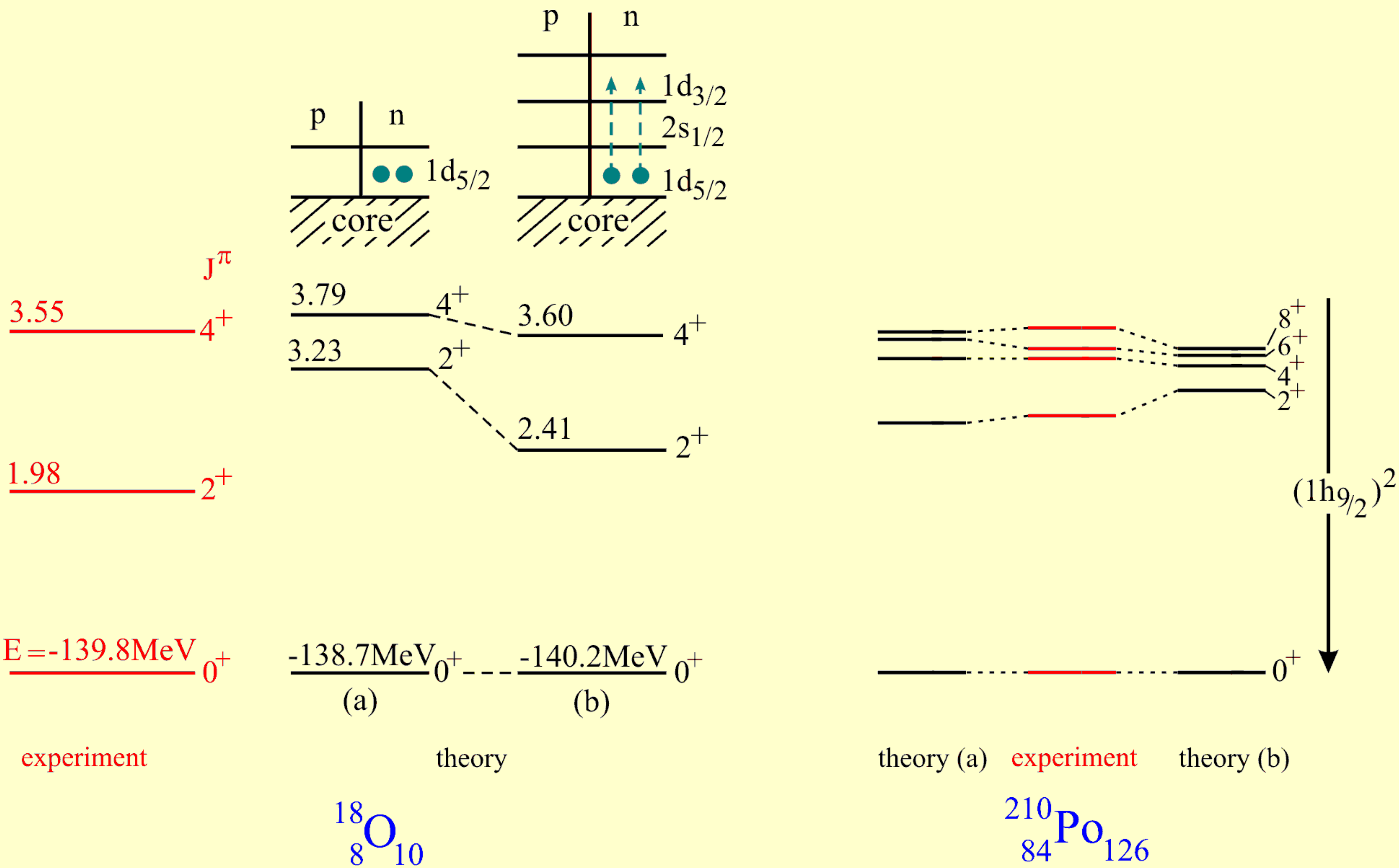
$\Delta E_{(j_a j_b)J}$ $\qquad$ $\Delta E_{(j)^2 J}$

Radial overlap of s.p. orbitals

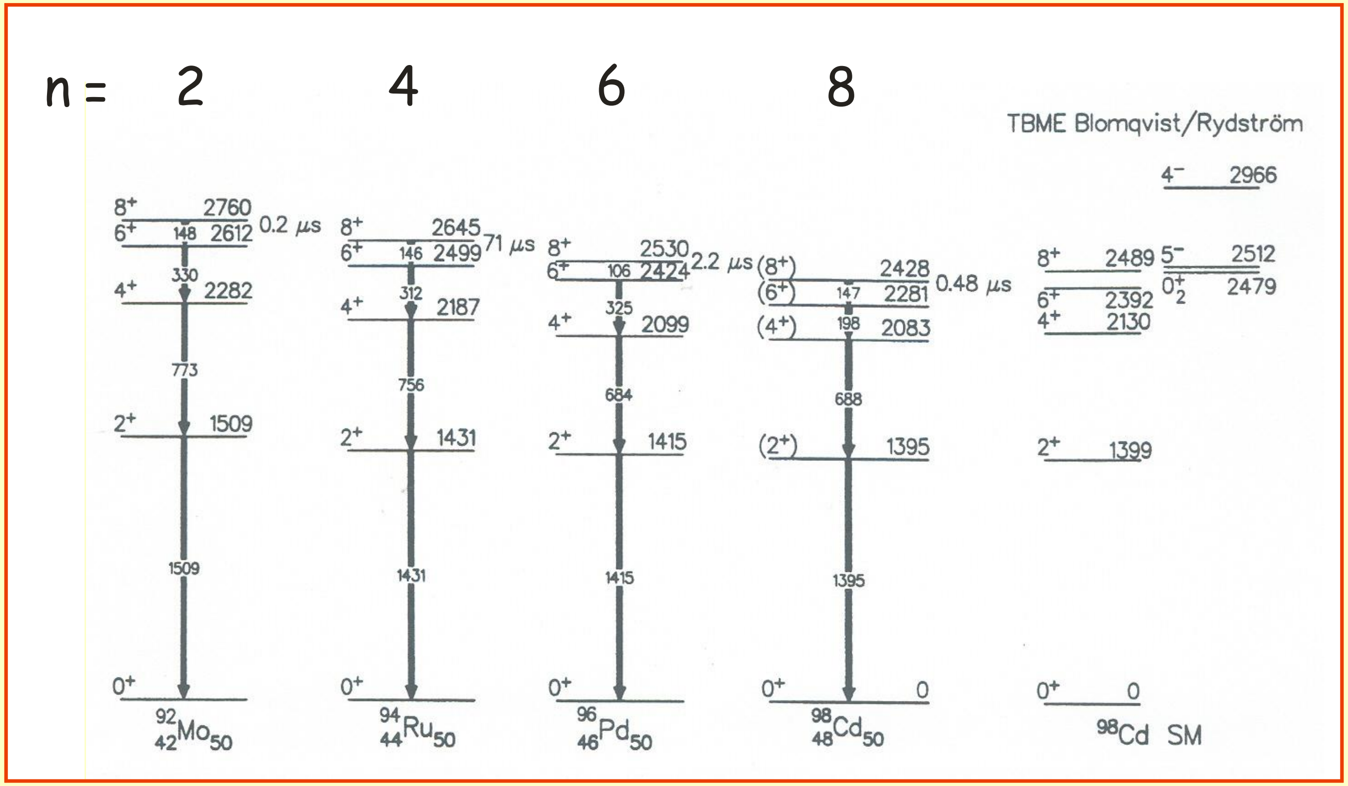
Use schematic nucleon-nucleon interaction: purely central.

$$V(|\vec{r}_1 - \vec{r}_2|) = \sum_k v_k(r_1, r_2) P_k(\cos\theta_{12}) \leftarrow \text{multipole expansion.}$$





p
n
1d$_{5/2}$
core
p
n
1d$_{3/2}$
2s$_{1/2}$
1d$_{5/2}$
core
J^π
3.55
4^+
1.98
2^+
E =-139.8MeV
0^+
3.79
4^+
3.23
2^+
-138.7MeV
0^+
(a)
3.60
4^+
2.41
2^+
-140.2MeV
0^+
(b)
experiment
theory
$^{18}_{8}O_{10}$
8^+
6^+
4^+
2^+
0^+
$(1h_{9/2})^2$
theory (a)
experiment
theory (b)
$^{210}_{84}Po_{126}$



N=50 nuclei

Filling the $1g_{9/2}$ orbital

$(1g_{9/2})^{-2}$

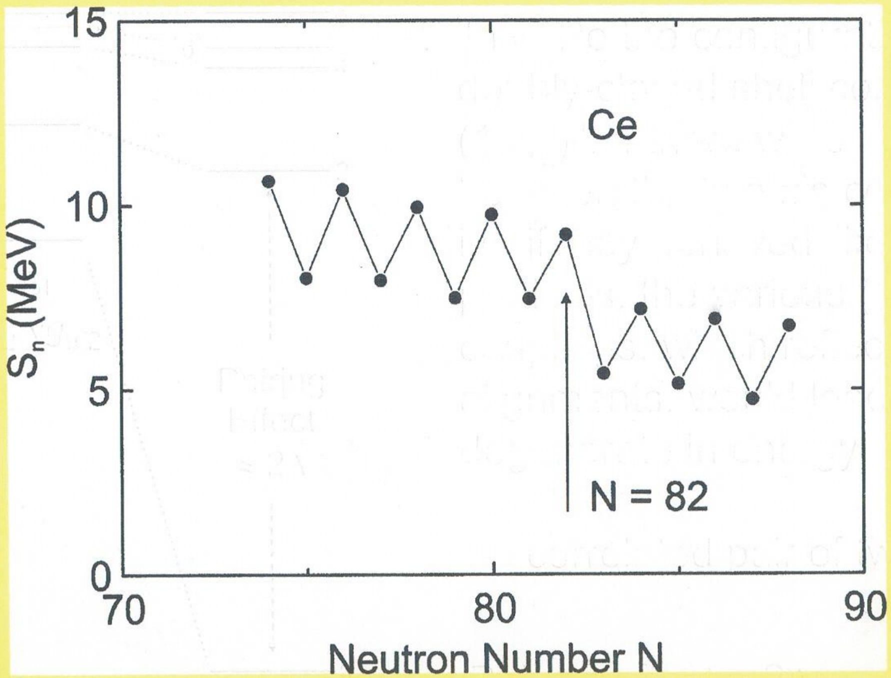
CONFIGURATION

$(2p_{1/2})^{-2}$

$(2p_{1/2}^{-1}\ 1g_{9/2})$

(i) Odd-even effect: mass of an odd-even nucleus is larger than the mean of adjacent two even-even nuclear masses → shows up in S_n and S_p for all nuclei.

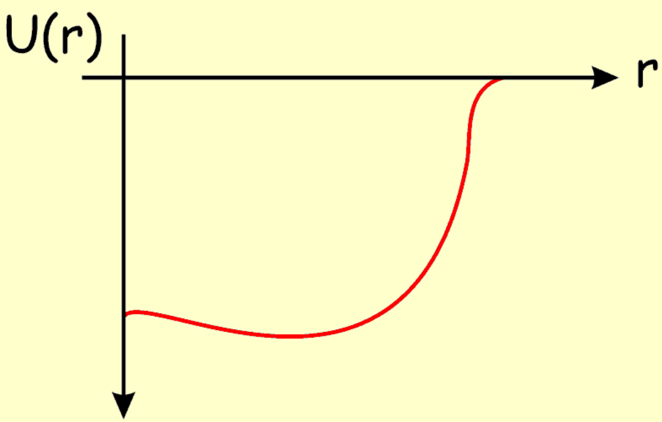
Example: $S_n = BE(A,Z) - BE(A\text{-}1,Z)$ of Ce nuclei



- Behavior points towards pair formation of nucleons.

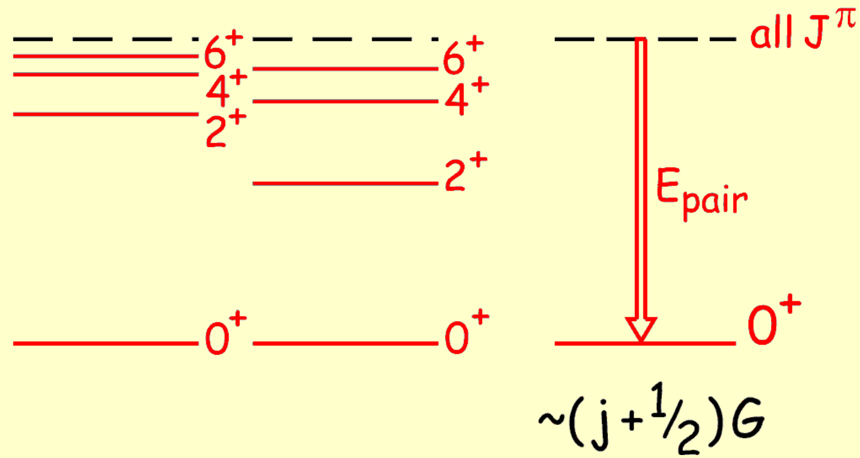
SITUATION

1. There exsist a well-defined average field → independent particle motion of nucleons.

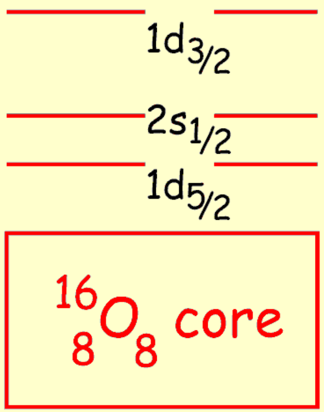


2. The two-body interaction (finite range, $\delta(\vec{r}_1 - \vec{r}_2)$,...) all show a "pairing" property

$$\langle j^2, JM | V_\delta | j^2, JM \rangle = A(j) \begin{pmatrix} j & j & J \\ 1/2 & -1/2 & 0 \end{pmatrix}^2$$



More realistic situation: consider "core" + 2 neutrons (protons)



ex. $^{18}_{8}O_{10}$

$$\left.\begin{aligned} |\psi_1^{(0)}\rangle &= |(1d_{5/2})^2; 0^+\rangle \\ |\psi_2^{(0)}\rangle &= |(2s_{1/2})^2; 0^+\rangle \\ |\psi_3^{(0)}\rangle &= |(1d_{3/2})^2; 0^+\rangle \end{aligned}\right\} \text{Basis}$$

Expand wave functions in this basis

$$|\psi_p\rangle = \sum_{k=1,..,3} a_{kp} |\psi_k^{(0)}\rangle$$

$H = h_o(1) + h_o(2) + V(1,2) \rightarrow$ eigenvalue equation $H|\psi_p\rangle = E_p|\psi_p\rangle$

$$\sum_k [\, H_{\ell k} - E_p \delta_{\ell k}] a_{kp} = 0$$

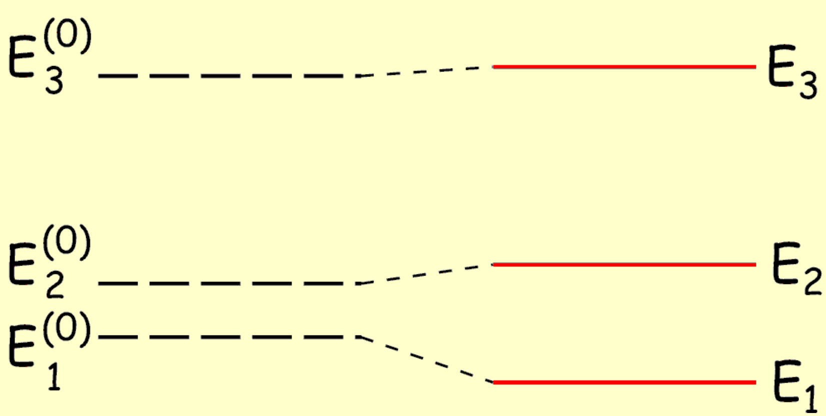
with $H_{\ell k} = \underbrace{E_k^{(0)}}_{\text{sum of s.p. energies}} \delta_{\ell k} + \langle \psi_\ell^{(0)} | V(1,2) | \psi_k^{(0)} \rangle$ ← 2-body matrix element

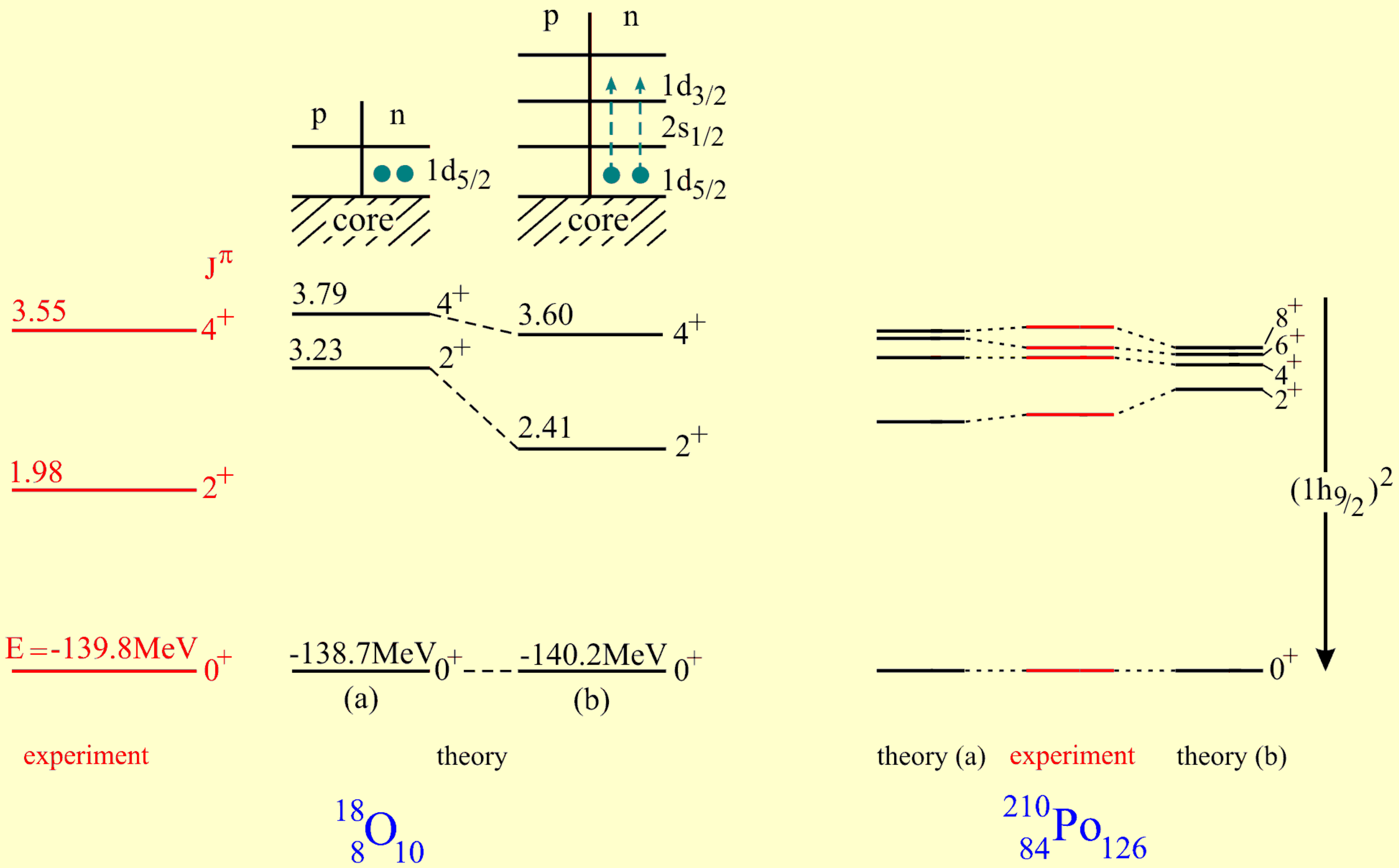
Eigenvalue secular equation $\rightarrow$ equivalent with diagonalization of the energy matrix $H_{\ell,k}$ (for $J^{\pi} = 0^{+}$ states)

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \Rightarrow \tilde{A}HA = E_p \mathbb{1}$$

with $H_{11} = 2\varepsilon_{1d_{5/2}} + \langle (1d_{5/2})^2; 0^+|V|(1d_{5/2})^2; 0^+\rangle$, H_{22}, H_{33} similar

$H_{12} = \langle (1d_{5/2})^2; 0^+|V|(2s_{1/2})^2; 0^+\rangle$, H_{13}, ... similar.





p
n
1d5/2
core
1d3/2
2s1/2
1d5/2
core
Jπ
3.55
4+
1.98
2+
E =-139.8MeV
0+
3.79
4+
3.23
2+
-138.7MeV
0+
(a)
3.60
4+
2.41
2+
-140.2MeV
0+
(b)
experiment
theory
18 8 O 10
8+
6+
4+
2+
0+
(1h9/2)2
theory (a)
experiment
theory (b)
210 84 Po 126

- Add more particles 2, 3, ..n one can use $n \rightarrow n+1 \rightarrow n+2$ coupling (c.f.p:coeff.of fractional parentage)

$$\psi(j^n \alpha; JM) = \sum_{\alpha', J'} [j^{n-1}(\alpha' J')j | \} j^n \alpha J] \, \psi(j^{n-1}(\alpha' J')j; JM)$$

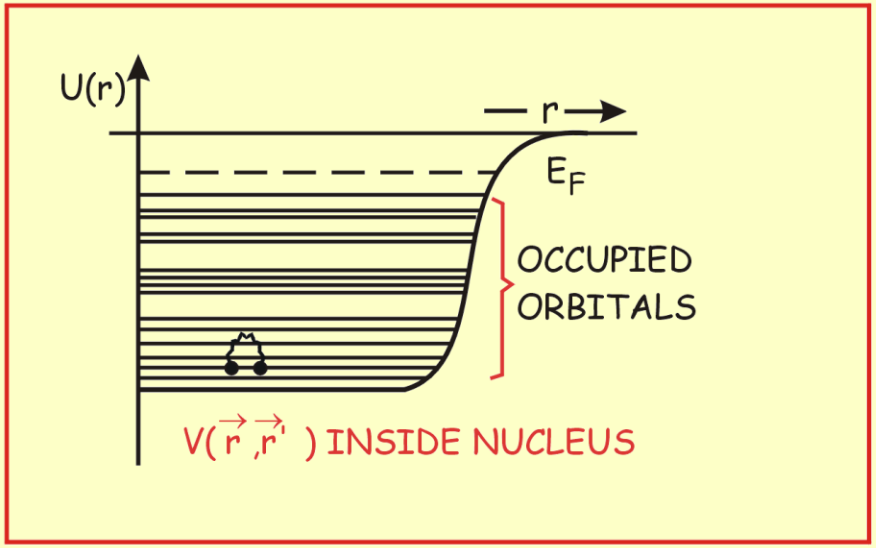
Alternative method: construct Slater determinant (M) and project out J values

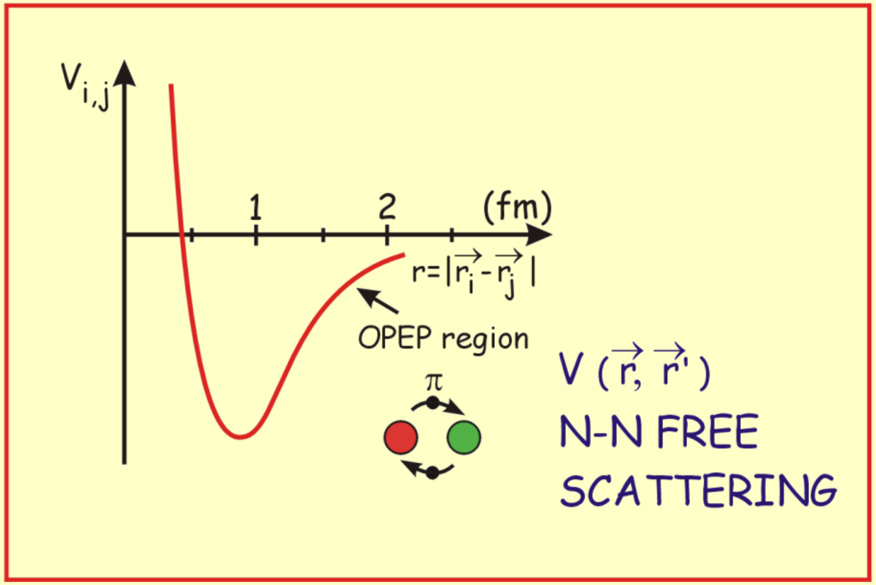
$$\psi(1,2,..A) = \frac{1}{\sqrt{A!}} \begin{bmatrix} \varphi_{\alpha_1}(\vec{r}_1) & \varphi_{\alpha_1}(\vec{r}_2) & \cdots & \varphi_{\alpha 1}(\vec{r}_A) \\ \varphi_{\alpha_2}(\vec{r}_1) & \varphi_{\alpha_2}(\vec{r}_2) & \cdots & \varphi_{\alpha 2}(\vec{r}_A) \\ \vdots & & & \\ \varphi_{\alpha_A}(\vec{r}_1) & \varphi_{\alpha_A}(\vec{r}_2) & \cdots & \varphi_{\alpha A}(\vec{r}_A) \end{bmatrix}$$

$\downarrow$

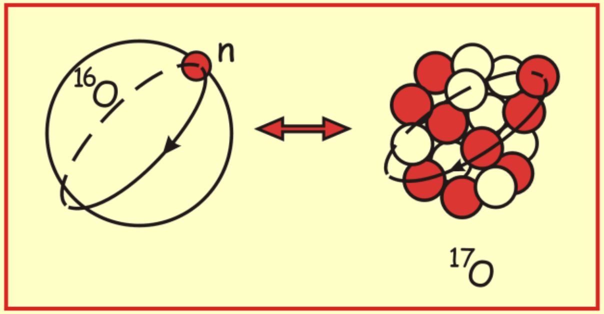
A very convient way for computing

Problems using realistic NN forces in nuclei





Concept of effective interaction (operators) active in finite space:



$$\Psi = \sum_i a_i \Psi_i \{17\text{-nucleon coordinates}\}.$$

$$\hat{O} = \sum_{i=1}^{17} \hat{O}\,(\vec{r}_i, \vec{\sigma}_i, \vec{\tau}_i).$$

More general

$(H_0 + V)\,\Psi = E\Psi \quad \Psi = \sum_{i=1}^{\infty} a_i \Psi_i^{(0)}$

FULL SPACE $(1,\ldots\infty)$

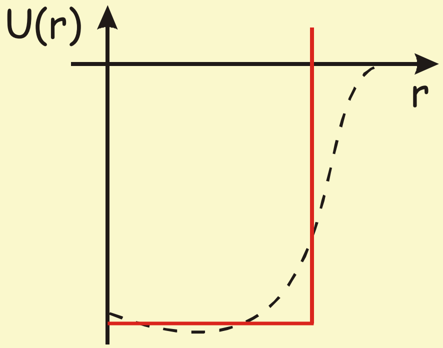
MODEL SPACE $(1,\ldots M)$ $\quad \Psi^M = \sum_{i=1}^{M} a_i \Psi_i^{(0)}$

$$\boxed{\text{IMPLICIT EQ. } \langle \Psi^M | H^{eff} | \Psi^M \rangle = E}$$

Above results can be obtained approximately using

- Square-well potential + strong spin-orbit term

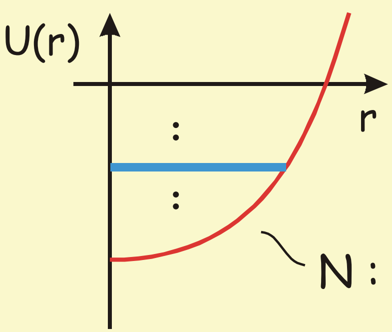
$\rightarrow J_{l+1/2}(kr)$



ex. N = 4

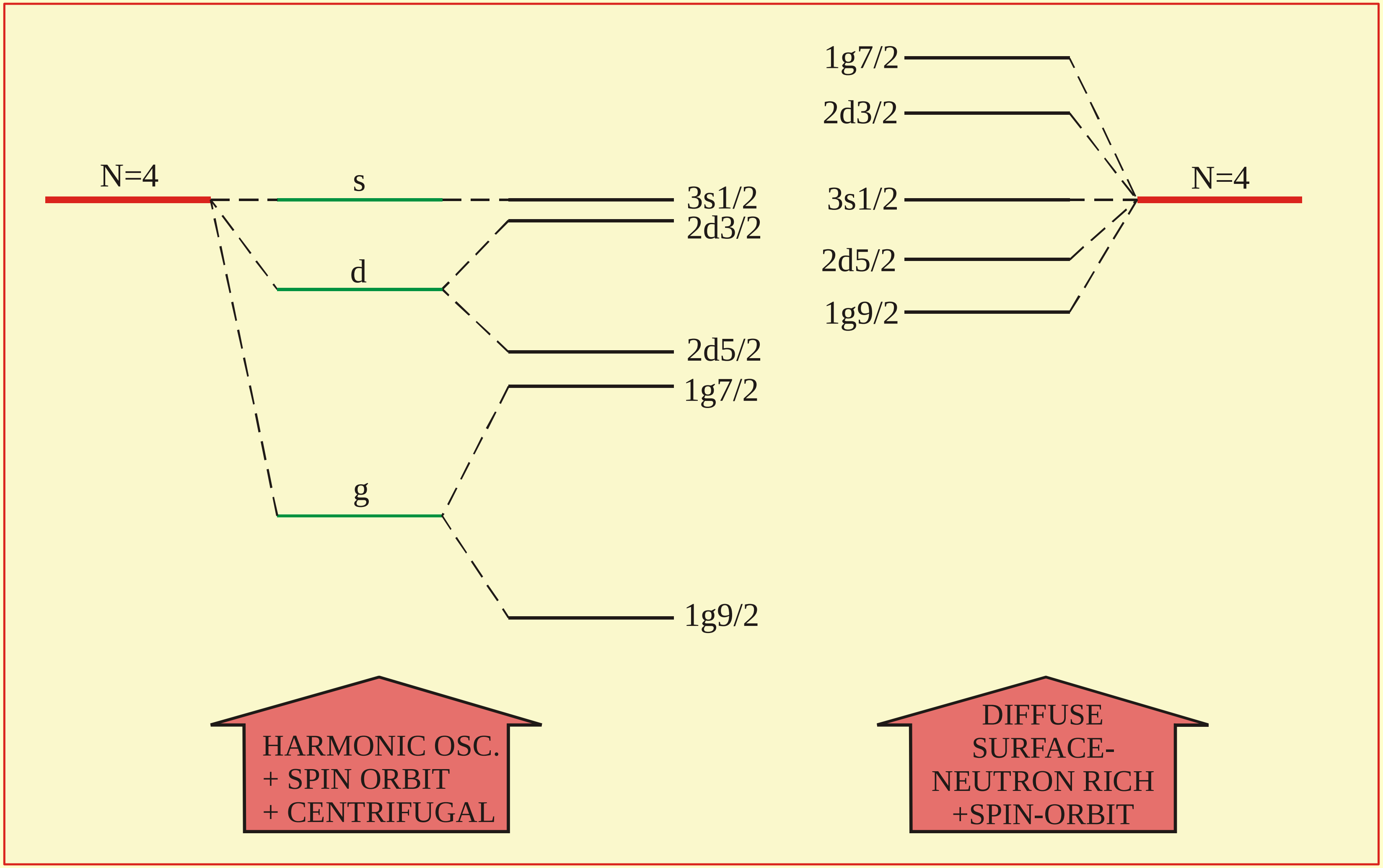
s l=0 (100)
d l=2 (93)
g l=4 (75.5)

- Harmonic oscillator potential + orbital + spin-orbit term



$$\rightarrow (\nu r)^l e^{\frac{-\nu^2 r^2}{2}} L_{n-1}^{l+1/2}(\nu^2 r^2)$$

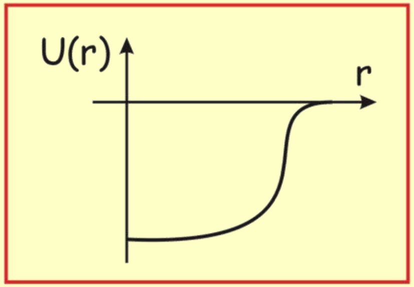
$$\left(\nu = \sqrt{\frac{m\omega}{\hbar}}\right)$$


N=4
s
d
g
3s1/2
2d3/2
2d5/2
1g7/2
1g9/2
HARMONIC OSC.
+ SPIN ORBIT
+ CENTRIFUGAL
1g7/2
2d3/2
3s1/2
2d5/2
1g9/2
N=4
DIFFUSE
SURFACE-
NEUTRON RICH
+SPIN-ORBIT

SITUATION:

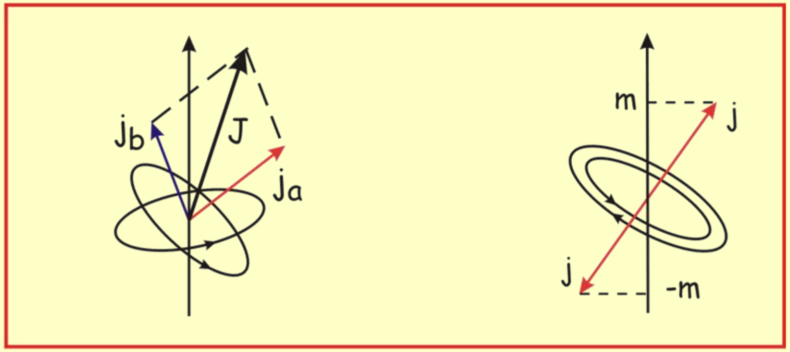
- There exists a well-defined mean field

  $\Rightarrow$ independent-particle motion



- There are strong two-nucleon correlations

  $\Rightarrow$ nucleon pair formation



$|j_a - j_b| \leq J \leq j_a + j_b$ $\qquad$ $J = 0, 2, 4, \ldots 2j-1$

$\Delta E_{(j_a j_b)J}$ $\qquad$ $\Delta E_{(j)^2 J}$

Radial overlap of s.p. orbitals

Many-body Hamiltonian-Nuclear mean-field

- Start from many-body Hamiltonian

$$H=\sum_{i=1}^{A}\frac{\vec{p}_i^{\,2}}{2m_i}+\frac{1}{2}\sum_{i,j=1}^{A}V_{ij}\,(|\vec{r}_i-\vec{r}_j|)$$

- Introduction of mean-field $U(r_i)$

$$H=\underbrace{\sum_{i=1}^{A}\left(\frac{\vec{p}_i^{\,2}}{2m_i}+U\,(r_i)\right)}_{h_o(i)}+\underbrace{\sum_{i<j=1}^{A}V_{ij}\,(|\vec{r}_i-\vec{r}_j|)-\sum_{i=1}^{A}U(ri)}_{\text{Residual interaction}}$$

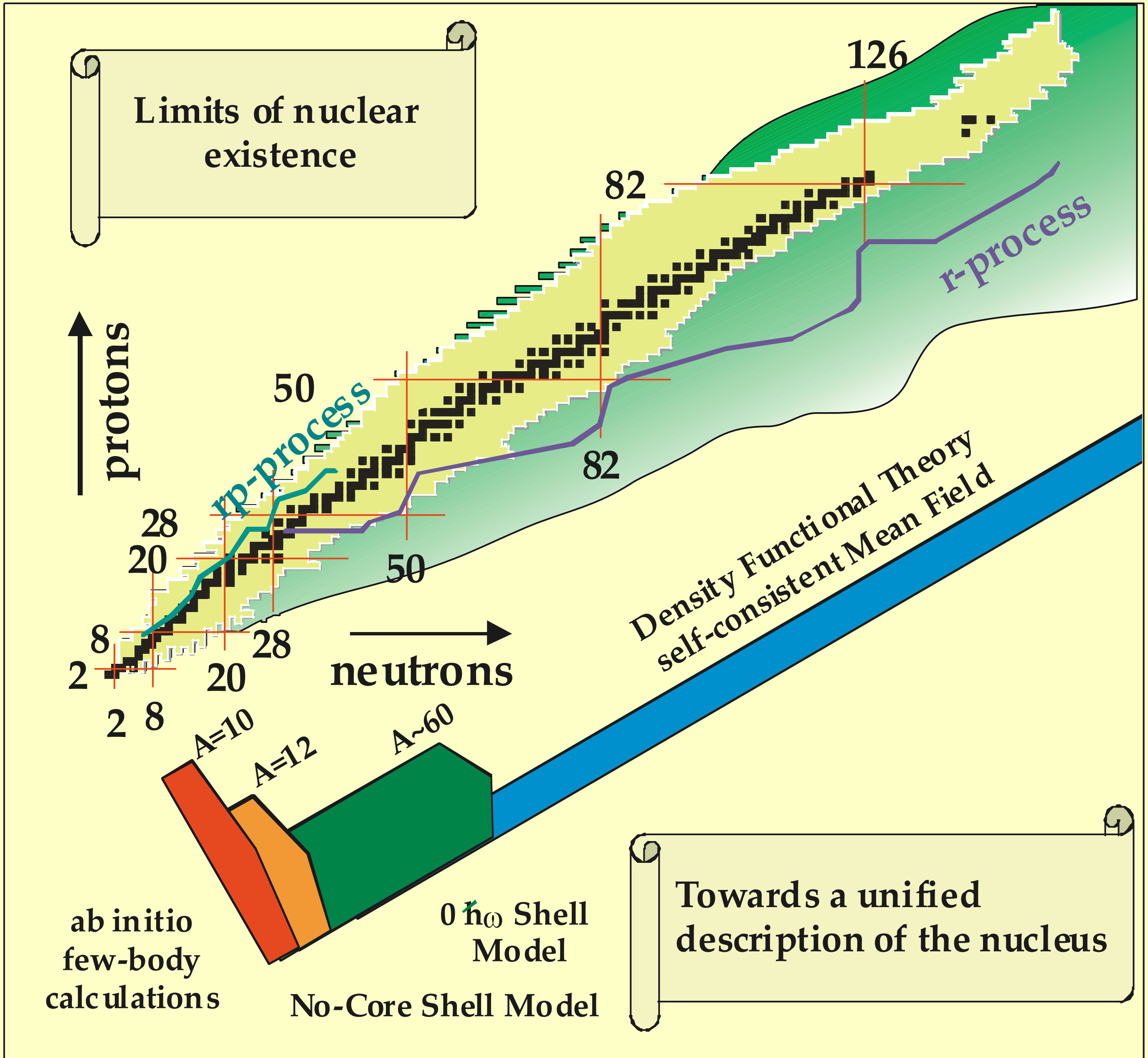
$$h_o\varphi_\alpha(\vec{r}_i)=\varepsilon_a\varphi_\alpha(\vec{r}_i)\qquad\Rightarrow\qquad\{\varphi_\alpha(\vec{r}_i),\ \varepsilon_a\}$$

Spherical field

$\alpha\equiv\{n_a, l_a, j_a, m_a\}$

Single-particle wave function,

Single-particle energy


Limits of nuclear existence
protons
neutrons
2
8
20
28
50
82
126
2
8
20
28
50
82
rp-process
r-process
Density Functional Theory
self-consistent Mean Field
A=10
A=12
A~60
ab initio few-body calculations
0 ħω Shell Model
No-Core Shell Model
Towards a unified description of the nucleus

- A=18, two-particle problem with ^{16}O core
  - Two protons: ^{18}Ne (T=1)
  - One Proton and one neutron: ^{18}F (T=0 and T=1)
  - Two neutrons: ^{18}O (T=1)

Homework:

Part 1

- How many states for each J_z? How many states of each J and T?

Part 2

- What are the energies of the three 0^+ states in ^{18}O?

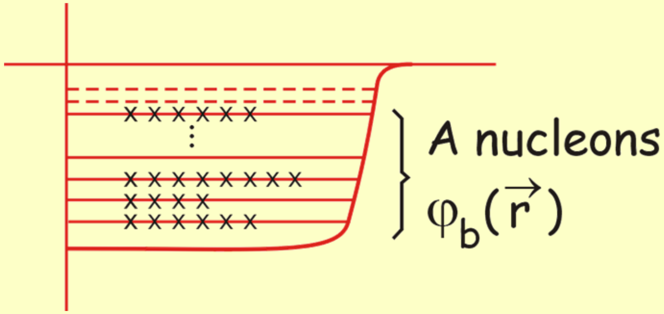
$$\varepsilon_{0d_{5/2}} = -3.94780$$
$$\varepsilon_{1s_{1/2}} = -3.16354$$
$$\varepsilon_{0d_{3/2}} = 1.64658$$

$$\langle 0d_{5/2}0d_{5/2}; J=0, T=1 | V | 0d_{5/2}0d_{5/2}; J=0, T=1 \rangle = -2.8197$$
$$\langle 0d_{5/2}0d_{5/2}; J=0, T=1 | V | 0d_{3/2}0d_{3/2}; J=0, T=1 \rangle = -3.1856$$
$$\langle 0d_{5/2}0d_{5/2}; J=0, T=1 | V | 1s_{1/2}1s_{1/2}; J=0, T=1 \rangle = -1.0835$$
$$\langle 1s_{1/2}1s_{1/2}; J=0, T=1 | V | 1s_{1/2}1s_{1/2}; J=0, T=1 \rangle = -2.1246$$
$$\langle 1s_{1/2}1s_{1/2}; J=0, T=1 | V | 0d_{3/2}0d_{3/2}; J=0, T=1 \rangle = -1.3247$$
$$\langle 0d_{3/2}0d_{3/2}; J=0, T=1 | V | 0d_{3/2}0d_{3/2}; J=0, T=1 \rangle = -2.1845$$

Hartree-Fock method

$$U(\vec{r}) = \int \rho(\vec{r}')\, V(\vec{r}, \vec{r}')\, d\vec{r}'$$

$$\rho(\vec{r}') = \sum_{b\{occ.\}} |\varphi_b(\vec{r}')|^2$$



Solve H. F. equations in self-consistent way, starting from $V(\vec{r}, \vec{r}')$ and initial guess for $\varphi_i(\vec{r})$

$$-\frac{\hbar^2}{2m} \Delta_i \varphi_i(\vec{r}) + U(\vec{r}_i)\varphi_i(\vec{r}) = \varepsilon_i \varphi_i(\vec{r})$$

Direct (Hartree) term

$$-\frac{\hbar^2}{2m} \Delta_i \varphi_i(\vec{r}) + \sum_{b\{occ\}} \int \varphi_b^*(\vec{r}')\; V(\vec{r}, \vec{r}')\; \varphi_b(\vec{r}')\; \varphi_i(\vec{r})\, d\vec{r}'$$

Exchange(Fock)term

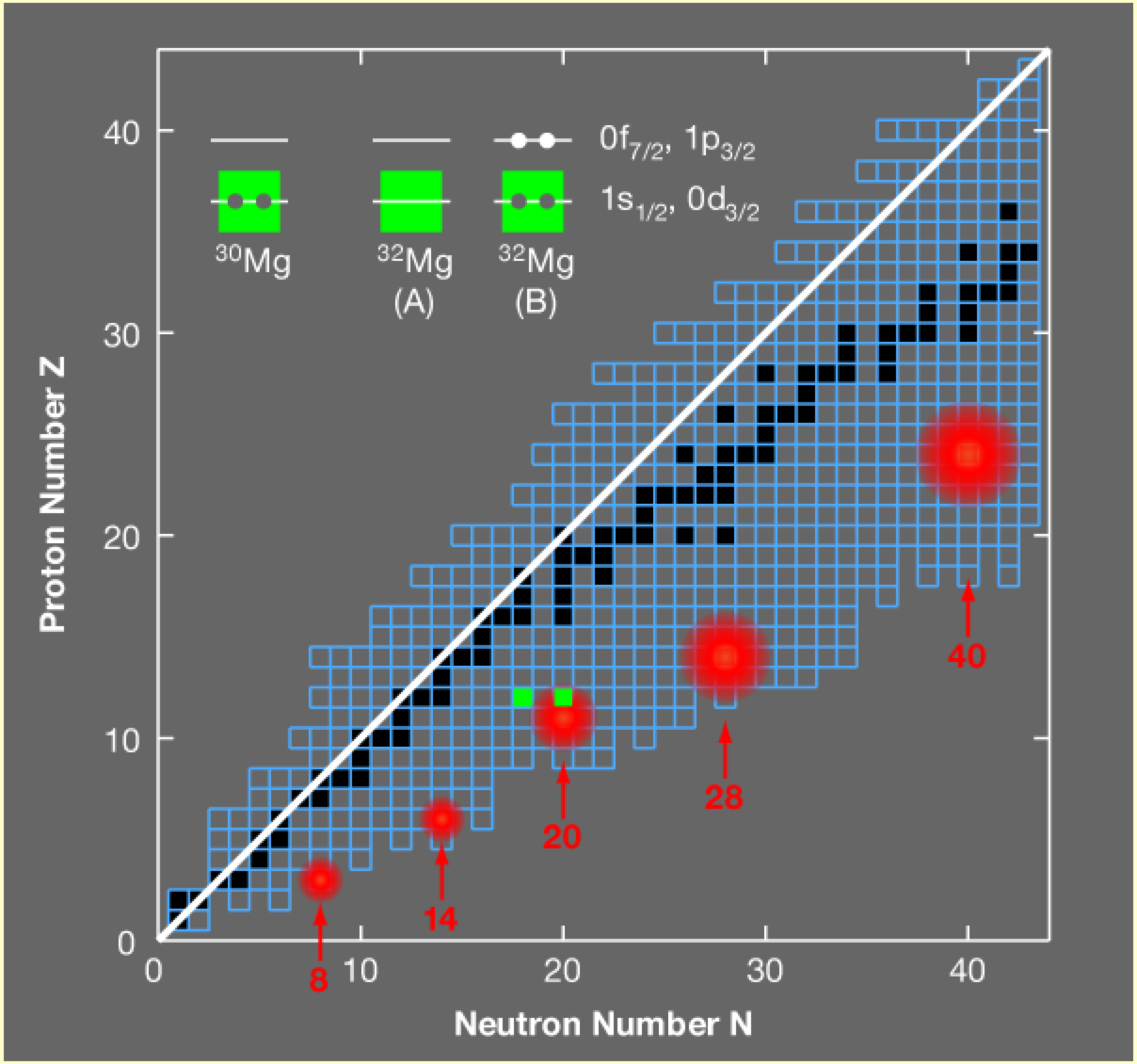
$$- \sum_{b\{occ\}} \int \varphi_b^*(\vec{r}')\; V(\vec{r}, \vec{r}')\; \varphi_b(\vec{r})\, \varphi_i(\vec{r}')\, d\vec{r}' = \varepsilon_i \varphi_i(\vec{r})$$

Iterate equations $\Rightarrow$ convergence

- **Total angular momentum, and isospin;** $\chi^{T}_{\frac{1}{2}s_z}$
- **Anti-symmetrized, two particle, jj-coupled wave function**

$$\psi^{j_1 j_2}_{JMTT_z} = \left\{ \left[\varphi_{n_1 l_1 j_1}(\vec{r}_1) \otimes \varphi_{n_2 l_2 j_2}(\vec{r}_2) \right]^{JM} + (-1)^{j_1+j_2+J+T} \left[\varphi_{n_2 l_2 j_2}(\vec{r}_1) \otimes \varphi_{n_1 l_1 j_1}(\vec{r}_2) \right]^{JM} \right.$$
$$\left[\chi^{T}_{\frac{1}{2}}(1) \otimes \chi^{T}_{\frac{1}{2}}(2) \right]^{TT_z} \Big/ \sqrt{2(1+\delta_{12})}$$

— **Note $J+T$=odd if the particles occupy the same orbits**

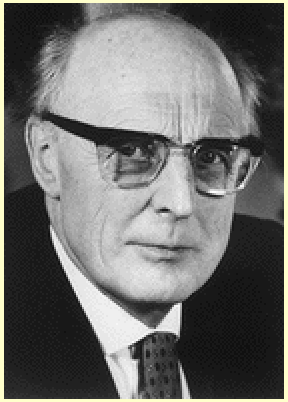


0f7/2, 1p3/2
1s1/2, 0d3/2
30Mg
32Mg (A)
32Mg (B)
Proton Number Z
Neutron Number N
0
10
20
30
40
8
14
20
28
40

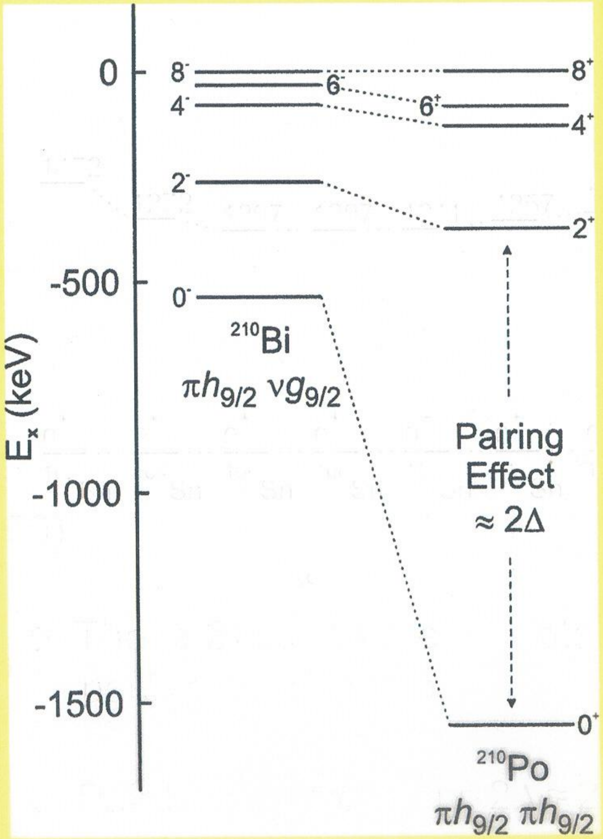
Idea of shell-model:

- M. Goeppert-Mayer (1949)
- H. Jensen, O. Haxel and H. E. Suess (1949)

$$U(r)=\frac{1}{2}\, m\omega^2 r^2+\alpha\, \vec{\ell}.\vec{\ell}+\beta\, \vec{\ell}.\vec{s}$$



(ii) Energy spectra for nuclei near closed shells (A ± 2, A ± 4) show a clear gap for the 0^+ g.s.
Example: ${}^{210}_{84}Po_{126}$ and ${}^{210}_{83}Bi_{127}$ adjacent to ${}^{208}_{82}Pb_{126}$



- In ^{210}Po the configuration outside the doubly-closed shell core of ^{208}Pb is $(1h_{9/2})^2$. If there were no interaction between the two π's constituting the pair, i.e. if they behaved like independent particles, the various $(1h_{9/2})^2$ spin couplings, which reflect the orbital alignments, would lead to states degenerate in energy.

  → correlated pair of two π's

- Pairing effect ≈ 2Δ